The BCS Energy Gap

BCS effective Hamiltonian in the grand canonical ensemble:

$$H - \mu N = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \frac{g}{V} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'+\boldsymbol{q}\downarrow} c_{\boldsymbol{k}'\uparrow}$$

where $\xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$, V is the system volume and $g > 0$ (coupling strength).

Suppose the ground state $|\Omega_s\rangle$ is characterised by a (real) macroscopic number Δ of cooper pairs:

$$\Delta = \frac{g}{V} \sum_{\boldsymbol{k}} \langle \Omega_s | c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} | \Omega_s \rangle \to \Delta^* = \Delta = \frac{g}{V} \sum_{\boldsymbol{k}} \langle \Omega_s | c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} | \Omega_s \rangle$$

and assume small fluctuations about Δ (mean-field). Taking interaction terms involving pairs only,

$$\begin{aligned}\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} c^\dagger_{\boldsymbol{k}+\boldsymbol{q}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'+\boldsymbol{q}\downarrow} c_{\boldsymbol{k}'\uparrow} &\to \sum_{\boldsymbol{k},\boldsymbol{k}'} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \\ &= \left[\frac{V\Delta}{g} + \left(\sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} - \frac{V\Delta}{g}\right)\right] \left[\frac{V\Delta}{g} + \left(\sum_{\boldsymbol{k}'} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} - \frac{V\Delta}{g}\right)\right] \\ &\simeq \frac{V\Delta}{g} \sum_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} + \frac{V\Delta}{g} \sum_{\boldsymbol{k}'} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} - \left(\frac{V\Delta}{g}\right)^2\end{aligned}$$

So in this approximation

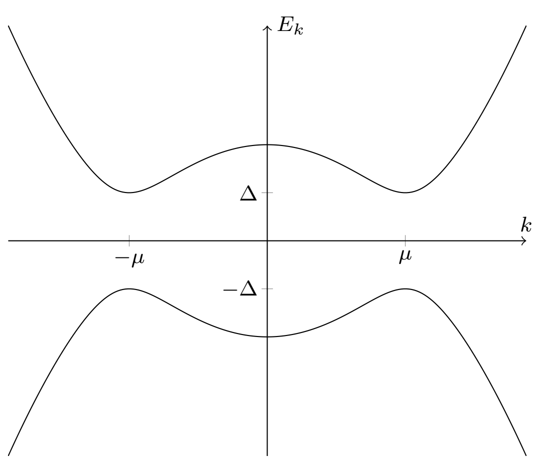
$$\begin{aligned}H - \mu N &= \sum_{\boldsymbol{k}} \left[\xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c_{\boldsymbol{k}\uparrow} + \xi_{-\boldsymbol{k}} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}\downarrow} - \Delta \left(c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} + c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}\right)\right] + \frac{V\Delta^2}{g} \\ &= \sum_{\boldsymbol{k}} \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} & c_{-\boldsymbol{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\boldsymbol{k}} & -\Delta \\ -\Delta & -\xi_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} + \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}} + \frac{V\Delta^2}{g}\end{aligned}$$

where we used the anticommutator $\{c_{-\boldsymbol{k}\downarrow}, c^\dagger_{-\boldsymbol{k}\downarrow}\} = 1$.

To determine the spectrum, solve the eigenvalue problem

$$\begin{vmatrix} \xi_{\boldsymbol{k}} - E_{\boldsymbol{k}} & -\Delta \\ -\Delta & -\xi_{\boldsymbol{k}} - E_{\boldsymbol{k}} \end{vmatrix} = 0 \Rightarrow E_{\boldsymbol{k}} = \pm\sqrt{(\epsilon_{\boldsymbol{k}} - \mu)^2 + \Delta^2}$$

Elementary excitations have minimum energy Δ:



Note H can be diagonalised by a Bogoliubov-Valatin transformation and a self-consistency equation obtained for Δ, leading to

$$\Delta(T=0) \simeq 2\hbar\omega_D e^{-1/(gN(0))} \quad \text{and} \quad k_B T_c = 2\pi^{-1} e^{\gamma} \hbar\omega_D e^{-1/(gN(0))}$$

and so the famous result

$$\frac{\Delta(T=0)}{k_B T_c} \simeq \pi e^{-\gamma} \approx 1.764$$

Here $N(0)$ is the density of states at the Fermi level, γ Euler's constant and ω_D the Debye frequency of the material (since $\omega_D \propto c \propto M^{-1/2}$, we also see the isotope effect). Refer to e.g. Tinkham [1] for the details.

Further Reading

[1] M. Tinkham Introduction to Superconductivity 2nd ed. (McGraw-Hill, New York, 1975).

BCS Theory including the variation calculation originally employed by BCS to determine the ground state, calculation of the gap relation and discontinuity in specific heat. Good coverage of superconductivity in general.

[2] A. Altland and B. D. Simons Condensed Matter Field Theory 2nd ed. (Cambridge University Press, Cambridge, 2010).

BCS in the field theory framework.

[3] J. F. Annett Superconductivity, Superfluids and Condensates (Oxford University Press, Oxford, 2004).

Accessible introduction to BCS and Ginzburg-Landau theory.

[4] A. L. Fetter and J. D. Walecka Quantum Theory of Many-Particle Systems (McGraw-Hill, San Francisco, 1971).

Advanced many-particle theory with application to superconductivity.

[5] P. W. Anderson Science **235** 1196 (1987)

Anderson's original paper on the resonating valence bond state in La_2CuO_4 and superconductivity.

[6] P. A. Lee, N. Nagaosa, and X.-G Wen Rev. Mod. Phys **78** 17 (2006)

Review of high temperature superconductivity by proponents of RVB.